Kaplansky Density Theorem

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In the sequel, H will be a fixed Hilbert space and $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H.

Let A be a C^*-subalgebra of $\mathcal{B}(H)$.

Abbreviations :

w.o. - weak-operator

s.o. - strong-operator

$(A)_1 := \{x \in A : \|x\| \leq 1\}$ (the closed unit ball of A)

$A_{sa} = \{x \in A : x^* = x\}$ (the set of self-adjoint operators in A)

We denote the closure of A in the s.o. topology of A by $\overline{A}^{SOT}$.

Recall

The **strong operator topology** on $\mathcal{B}(H)$ is a locally convex topology generated by the seminorms $\|.\|x, x \in H$ where

$$\|T\|_x = \|Tx\|, T \in \mathcal{B}(H).$$

The **weak operator topology** on $\mathcal{B}(H)$ is a locally convex topology generated by the seminorms $\|.\|_{x,y}, x, y \in H$ where

$$\|T\|_{x,y} = |\langle Tx, y\rangle|, T \in \mathcal{B}(H).$$

Useful Facts

(1) Weak topology $\subseteq$ strong topology.
The inclusion is strict when $\dim H = \infty$.

(2) If $x_\alpha \to x$ in s.o. topology, then $x_\alpha \to x$ in the w.o. topology.

(3) $x \mapsto x^*$ is w.o. continuous on $\mathcal{B}(H)$
$x \mapsto Re\ x$ is w.o. continuous on $\mathcal{B}(H)$

(4) If S is a convex subset of $\mathcal{B}(H)$,
then

$$\overline{S}^{WOT} = \overline{S}^{SOT}.$$

(5) If f is a real-valued continuous function on $\mathbb{R}$ which vanishes at ∞,
then $T \mapsto f(T)$ is s.o. continuous on $\mathcal{B}(H)_{sa}$.

Kaplansky Density Theorem

Let A be a C^*-subalgebra of $\mathcal{B}(H)$ and let $M = \overline{A}^{SOT}$. Then

(a) $(M_{sa})_1 = \overline{(A_{sa})_1}^{SOT}$.

That is, the closed unit ball of M_{sa} is the s.o. closure of the closed unit ball of A_{sa}. In other words, the closed unit ball of A_{sa} is s.o. dense in the closed unit ball of M_{sa}. The above statement is true for positive and unitary operators.

(b) $M_{sa} = \overline{A}_{sa}^{SOT}$.

The above expression is true for positive and unitary operators.

(c) $(M)_1 = \overline{(A)_1}^{SOT}$.

Proof : $(M_{sa})_1 = \overline{(A_{sa})_1}^{SOT}$

(a) We have $M = \overline{A}^{SOT} \to (1)$

Let $x \in (M_{sa})_1$.

By (1), $\exists$ a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$ in the s.o. topology.

Since $x_\alpha \to x$ in s.o. topology, $x_\alpha \to x$ in w.o. topology.

Let $y_\alpha = \frac{x_\alpha + x_\alpha^*}{2}$. Then $y_\alpha \in A_{sa}$.

Since $x \mapsto x^*$ is w.o. continuous on $\mathcal{B}(H)$,

$x_\alpha^* \to x^*$, hence $\{y_\alpha\}$ in A_{sa} converges to $\frac{x+x^*}{2} = x$ in the w.o. topology.

Since A_{sa} is convex, $\overline{A}_{sa}^{WOT} = \overline{A}_{sa}^{SOT}$.

Hence $\exists$ a net $\{z_\alpha\}$ in A_{sa} such that $z_\alpha \to x$ in the s.o. topology.

Consider the real-valued continuous function f on $\mathbb{R}$ defined by

$$f(t) = \begin{cases} t & |t| \leq 1 \\ \frac{1}{t} & |t| \geq 1 \end{cases}$$

$\therefore x \mapsto f(x)$ is s.o. continuous on $\mathcal{B}(H)_{sa}$.
As $z_\alpha \in A_{sa}$ and $z_\alpha \xrightarrow{SOT} x$, $f(z_\alpha) \xrightarrow{SOT} f(x)$.
However, x is self-adjoint and $\|x\| \leq 1$, so $\sigma(x) \in [-1, 1]$, so $f|_{\sigma(x)} = t$ and $f(x) = x$ by the functional calculus.
Moreover $\overline{f} = f$ and $\|f\|_\infty \leq 1$, so $f(z_\alpha)^* = f(z_\alpha)$ and $\|f(z_\alpha)\| \leq 1$ for all α.
That is, $f(z_\alpha) \in (A_{sa})_1$ and $f(z_\alpha) \xrightarrow{SOT} f(x) = x$, so $x \in \overline{(A_{sa})_1}^{SOT}$.

$$\therefore (M_{sa})_1 = \overline{(A_{sa})_1}^{SOT}.$$

Proof : $M_{sa} = \overline{A}_{sa}^{SOT}$

(b) Let $x \in M_{sa}$. Since $M = \overline{A}^{SOT}$, a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$ in the s.o. topology.
$x_\alpha \xrightarrow{SOT} x \Rightarrow x_\alpha \xrightarrow{WOT} x \Rightarrow Re\ x_\alpha \xrightarrow{WOT} Re\ x = x$ ($\because x \mapsto Re\ x$ is w.o. continuous). Since A is C^*-subalgebra, $Re\ x_\alpha \in A_{sa}$.
Also we have $Re\ x_\alpha \xrightarrow{WOT} x$. As A_{sa} is convex, $x \in \overline{A_{sa}}^{WOT} = \overline{A_{sa}}^{SOT}$, so $\exists$ $\{z_\alpha\}$ in A_{sa} such that

$$z_\alpha \xrightarrow{SOT} x.$$

$\therefore M_{sa} = \overline{A_{sa}}^{SOT}$.

Proof : $(M)_1 = \overline{A_1}^{SOT}$.

(c) Let $x \in (M)_1$. Define $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(M)$. Then $\tilde{x}$ is self-adjoint.

Since $\|\tilde{x}\| = \sup_{\|(f,g)\| \leq 1} \|\tilde{x}(f,g)\| = \sup_{\|(f,g)\| \leq 1} (\|xg\|^2 + \|xf\|^2)^{\frac{1}{2}}$
$\leq \sup_{\|(f,g)\| \leq 1} (\|g\|^2 + \|f\|^2)^{\frac{1}{2}} = 1$, we have $\|\tilde{x}\| \leq 1$.

Since $M = \overline{A}^{SOT}$, $M_2(M) = \overline{M_2(A)}^{SOT} \rightarrow (2)$ (Exercise).
From (2) and part (a), $\exists$ $\{\tilde{y}_\alpha\}$ in $(M_2(A)_{sa})_1$ such that $\tilde{y}_\alpha \xrightarrow{SOT} \tilde{x}$.
Since $\{\tilde{y}_\alpha\}$ is self-adjoint in $M_2(A)$, it is of the form

$$\tilde{y}_\alpha = \begin{pmatrix} z_\alpha & x_\alpha \\ x_\alpha^* & w_\alpha \end{pmatrix} \qquad \text{for each } \alpha.$$

Then for each x_α is in A with $\|x_\alpha\| \leq 1$ and $x_\alpha \xrightarrow{SOT} x$ (Exercise).

Reference

Kehe Zhu, *An Introduction to Operator Algebras*, CRC Press, 1993.